Automated Reasoning for Experimental Mathematics Part II: the Erdős Discrepancy Conjecture

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Outline

- Part I: Automated Reasoning for Knots (computational topology)
- **Part II: Solution for the Erdős Discrepancy Problem, C=2 (combinatorial number theory)**
- Part III: Exploration of the Andrews-Curtis Conjecture (computational combinatorial group theory)

Part II is based on

(KL2014) Boris Konev, Alexei Lisitsa: A SAT Attack on the Erdős Discrepancy Conjecture. SAT 2014: 219-226

(KL2015) Boris Konev, Alexei Lisitsa: Computer-aided proof of Erdős discrepancy properties. Artif. Intell. 224: 103-118 (2015)

A very short version

The Erdős discrepancy conjecture is interesting (even for $C = 2$).

EDP2 can be settled by reduction to SAT.

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Longer version

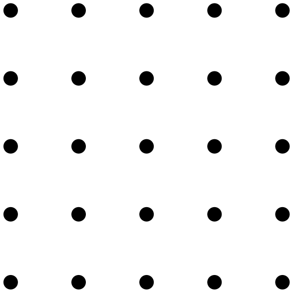
The Erdős discrepancy conjecture is interesting (even for $C = 2$).
EDP2 can be settled by reduction to SAT.

- Discrepancy Theory
- Erdős Discrepancy Conjecture
- SAT attack on the EDP
- Results and Perspectives

Discrepancy theory

Discrepancy theory is a branch of mathematics dealing with
inevitable irregularities of distributions

Combinatorial discrepancy



- A set U
- Set of subsets S

- A hypergraph $\mathcal{H} = (U, S)$
- Consider a colouring $c : U \to \{+1, -1\}$ of the elements of U in *blue* $(+1)$ and *red* (-1) colours;
- **Question:** Is there a colouring such that in every element of S colours are distributed uniformly or a **discrepancy** of colours is always inevitable?

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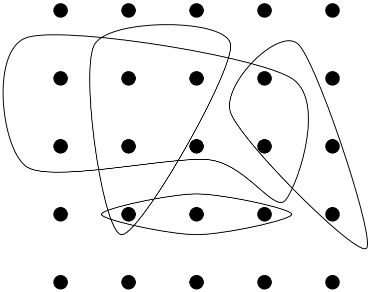

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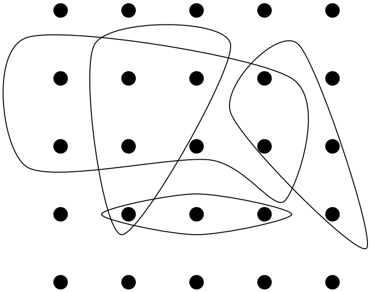

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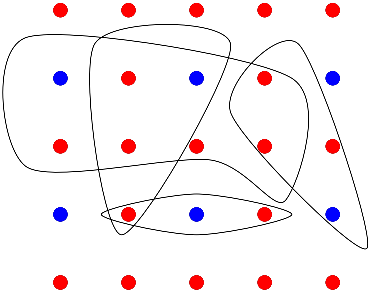

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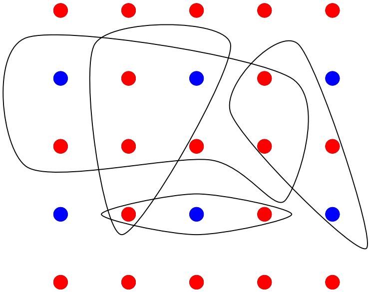

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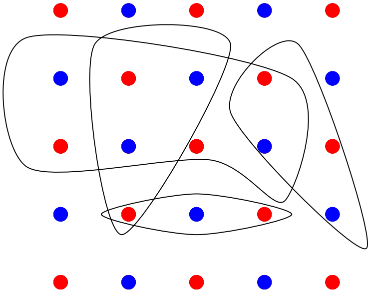

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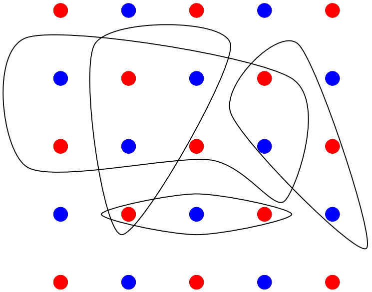

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Combinatorial number theory

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16	17	18	19	20
21	22	23	24	25

- $U \subseteq \mathbb{N}$
- S — 'arithmetically interesting' subsets of U

Theorem (Roth, 1964)

For $U_n = \{1, 2, \ldots, n\}$ and $S_n = \{(a \cdot i + b)\}$ the discrepancy grows at least as $\frac{1}{20} n^{1/4}$.

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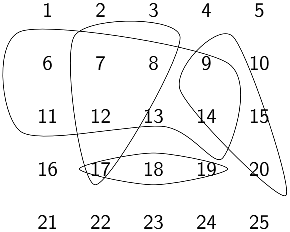

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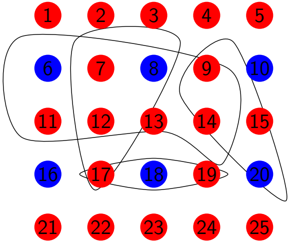

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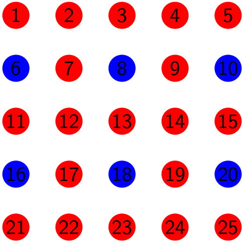

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What about **homogeneous** arithmetic progressions?

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Conjecture (Erdős, circa 1930)

For any $C > 0$ in any infinite ± 1 sequence (x_n) there exists a subsequence $x_d, x_{2d}, x_{3d}, \ldots, x_{kd}$ such that $|\sum_{i=1}^{k} x_{i \cdot d}| > C$.

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Known results on the discrepancy of $+1$ -1 sequences

- For random ± 1 sequences the discrepancy grows as $n^{1/2+o(1)}$ (folklore?);
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+	−	−	+	−	+	_	−	_	_	_	_
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+	−	−	+	−	+	+	−	_	+	_	_	$d = 5$
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But then $x_3 + x_6 + x_9 + x_{12} = -1 + 1 - 1 - 1 = -2$

Polymath and the EDP

- A 2009-2010 topic of Polymath — ‘massively collaborative maths’ project started and coordinated by T. Gowers
  - A computer attack
  - Discrepancy 2 sequences of length 1124 (backtracking search)

“. . . given how long a finite sequence can be, it seems unlikely that we could answer this question just by a clever search of all possibilities on a computer. . . ”

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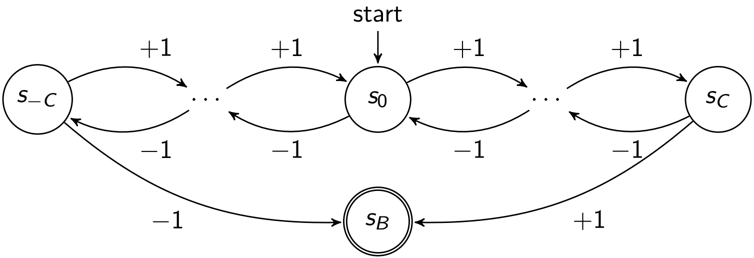
Our contribution

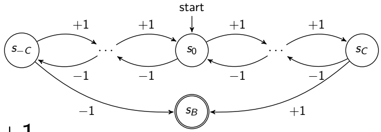
- There are ± 1 sequences of length 1160 and discrepancy 2,
- There are no ± 1 sequences of length 1161 (or more) and discrepancy 2.

Automata encoding of discrepancy conditions



- If for every $d : 1 \leq d \leq \lfloor \frac{n}{C+1} \rfloor$ the automaton $\mathcal{A}_C$ *does not* accept the subsequence $x_d, x_{2d}, \dots, x_{kd}$, where $k = \lfloor \frac{n}{d} \rfloor$ then the discrepancy of the sequence $\bar{x}$ does not exceed C

SAT representation



- p_i is true if $\iff$ i-th letter is $+1$.
- $s_j^{(i,d)}$ is true $\iff$ $\mathcal{A}_C$ is in s_j having read first $(i-1)$ letters.

$$\phi_{(n,C,d)} = s_0^{(1,d)} \bigwedge_{i=1}^{\lfloor \frac{n}{d} \rfloor} \Bigg[\bigwedge_{-C \leq j < C} \left(s_j^{(i,d)} \wedge p_{i \cdot d} \to s_{j+1}^{(i+1,d)} \right) \wedge \bigwedge_{-C < j \leq C} \left(s_j^{(i,d)} \wedge \neg p_{i \cdot d} \to s_{j-1}^{(i+1,d)} \right) \wedge \left(s_C^{(i,d)} \wedge p_{i \cdot d} \to B \right) \wedge \left(s_{-C}^{(i,d)} \wedge \neg p_{i \cdot d} \to B \right) \Bigg]$$

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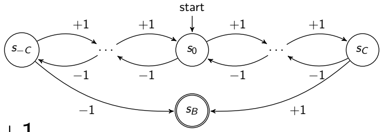

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Adequacy and correctness of encoding

Let

$$\phi_{(n,C)} = \neg B \wedge \bigwedge_{d=1}^{\lfloor \frac{n}{C+1} \rfloor} \phi_{(n,C,d)} \wedge \mathrm{frame}_{(n,C)},$$

where $\mathrm{frame}_{(n,C)}$ is a Boolean formula encoding that the automaton state is correctly defined.

Proposition

The formula $\phi_{(n,C)}$ is satisfiable if, and only if, there exists a ± 1 sequence $\bar{x} = x_1, \ldots, x_n$ of length n of discrepancy C. Moreover, if $\phi_{(n,C)}$ is satisfiable, the sequence $\bar{x} = x_1, \ldots, x_n$ of discrepancy C is uniquely identified by the assignment of truth values to propositions $p_1, \ldots p_n$.

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Disclaimer

In fact we have used more "economical" encoding, where a state s_i of automaton is encoded not by a separate propositional variable p_i, but by the propositional variables encoding i in binary

Experiments and Results

In our experiments we used

- the Lingeling SAT solver, the winner of the *SAT-UNSAT* category of the SAT'13 competition, and
- the Glucose solver version, the winner of the *certified UNSAT* category of the SAT'13 competition.

All experiments were conducted on PCs equipped with an Intel Core i5-2500K CPU running at 3.30GHz and 16GB of RAM.

Experiments and Results

- By iteratively increasing the length of the sequence, we establish precisely that the maximal length of a ± 1 sequence of discrepancy 2 is 1160.
- On our system it took Plingeling, the parallel version of the Lingeling solver, about 800 seconds to generate a sequence of discrepancy 2 and length 1160.
- On the other hand, when we increased the length of the sequence to 1161, Plingeling reported unsatisfiability.
- We also used Glucose: It took the solver about 21 500 seconds to compute a Delete Reverse Unit Propagation (DRUP) certificate of unsatisfiability ($\sim$ 13Gb).
- The certificate has been independently verified by the drup-trim tool

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- On the other hand, when we increased the length of the sequence to 1161, Plingeling reported unsatisfiability.
- We also used Glucose: It took the solver about 21 500 seconds to compute a Delete Reverse Unit Propagation (DRUP) certificate of unsatisfiability ($\sim$ 13Gb).
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- It is easy to check, either by a simple program, or even by hands that a 1160 sequence has discrepancy 2;
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EDP3 ?

We have applied the same methodology to the case C=3.

Proposition

There exists a sequence of length $\approx 14,000$ *of discrepancy* 3.

Life after SAT:

- Better SAT encodings
- Tuned search strategy
- 850Mb RUP certificate for EDP2

Life after SAT: Journal submission, CoRR abs/1405.3097

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$$x_{m \cdot n} = x_m \cdot x_n$$

- Longest completely multiplicative EDP3 sequences contains 127 645 elements
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Life after SAT, 2015

Terence Tao, September 2015

Proof of general case of EDP.

Conjecture (Erdős, circa 1930)

For any $C > 0$ in any infinite ± 1 sequence (x_n) there exists a subsequence $x_d, x_{2d}, x_{3d}, \ldots, x_{kd}$ such that $|\sum_{i=1}^{k} x_{i \cdot d}| > C$.

Not SAT solving or automated reasoning, but full power of Fields prize laureate's mind!

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- Reignited the debate on what a mathematical proof is
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A sequence of length 1160 and discrepancy 2

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